Generalized Cauchy–Schwarz Inequality

Vincent Tam

Notes for a journal article on this inequality

Review: AM–GM Inequality

For any positive real numbers

$a_1, \ldots, a_n$,

$$\underbrace{\left(\prod_{k=1}^{n} a_i\right)^{1/n}}_{\substack{\text{geometric mean} \\ \text{of } a_1,\ldots,a_n}} \leq \underbrace{\frac{1}{n}\left(\sum_{k=1}^{n} a_i\right)}_{\substack{\text{arithmetic mean} \\ \text{of } a_1,\ldots,a_n}}$$

Equality holds if and only if all

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Proof. (by replacement) If all a_i's are equal, then it's trivial. □

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$$\alpha = \underbrace{\mathrm{AM}(a_1, \ldots, a_n)}_{\text{arithmetic mean of } a_1, \ldots, a_n}.$$

□

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Then there exists $\underbrace{i \text{ and } j}_{\text{indices}}$ such that

$a_i < \alpha < a_j$.

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Then there exists $\underbrace{i \text{ and } j}_{\text{indices}}$ such that $a_i < \alpha < a_j$. ('α' comes from "<u>a</u>rithmetic mean") □

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$$\alpha = \underbrace{\mathrm{AM}(a_1, \ldots, a_n)}_{\text{arithmetic mean of } a_1,\ldots,a_n}.$$

Then there exists $\underbrace{i \text{ and } j}_{\text{indices}}$ such that $a_i < \alpha < a_j$.

$$\underbrace{a_i \longleftarrow \alpha}_{a_i \text{ replaced by } \alpha}, a_j \longleftarrow a_i + a_j - \alpha,$$

so after this replacement process,

$$\text{old AM} = \text{new AM}, \text{but}$$

$$\text{old GM} < \text{new GM because}$$

$$a_i a_j < \alpha(a_i + a_j - \alpha)$$

$$\Leftrightarrow (\alpha - a_i)(\alpha - a_j) < 0.$$

After this replacement, in the new AM we have at least one less number not equal to α,
This process can be repeated until all a_i's are equal (to α), then

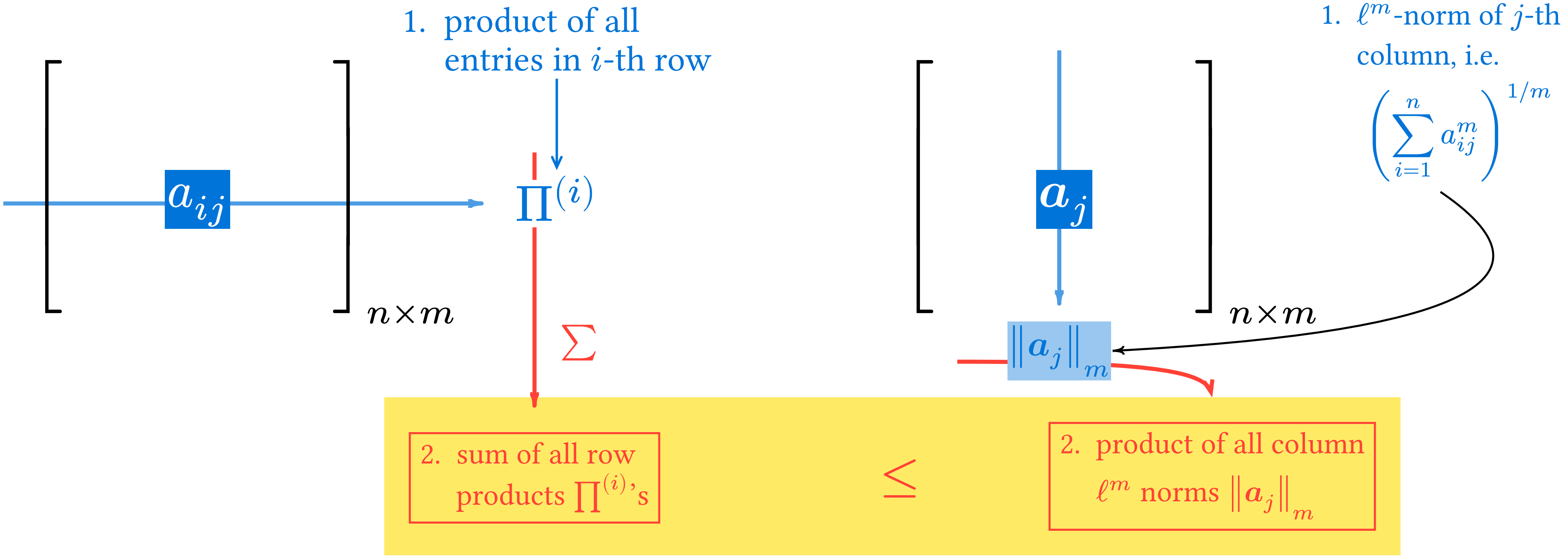
$$\text{final GM} = \alpha.$$

Hence

$$\text{initial AM} = \alpha = \text{final GM}$$
$$\geq \text{new GM} > \text{initial GM}.$$

$\square$

Generalized Cauchy–Schwarz Inequality



Question: How to remember this inequality?

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- '$\longrightarrow$' linked with '$\prod$' as 'H' in "Horizontal" looks like '$\coprod$ / $\prod$'
- '$\downarrow$' linked with '$\sum$' as two 'V''s ("Vertical") looks like '$\leq$ / $\vee$'

Orders of arrows in each side:

- LHS: natural reading order ("**less strange**")
  1. "left $\longrightarrow$ right" first
  2. then top $\downarrow$ bottom
- RHS: "**more strange**" reading order
  1. top $\downarrow$ bottom first
  2. then "left $\longrightarrow$ right"

Proof step 1: homogeneity on each column

Overall strategy is similar to the proof of Hölder's inequality.
Observe that in the target inequality

$$\sum_{i=1}^{n} \prod_{j=1}^{m} a_{ij} \leq \prod_{j=1}^{m} \left(\sum_{i=1}^{n} a_{ij}^{m} \right)^{1/m},$$

if we multiply (the entries of) the j-th column by a positive constant k (i.e. for each $i \in \{1, ..., n\}$ and a particular fixed $j \in \{1, ..., n\}$, $a_{ij} \longleftarrow ka_{ij}$), each side of the above inequality is also multiplied by k.

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WLOG (without loss of generality), we can assume that the j-th column is normalized, i.e. $\|\boldsymbol{a_j}\|_m = 1$.

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WLOG (without loss of generality), we can assume that the j-th column is normalized, i.e. $\left\|\boldsymbol{a_j}\right\|_m = 1$. The same goes for the remaining columns. Then the RHS becomes 1.

Proof step 2: AM–GM on each row product

$$\boxed{\prod_{j=1}^{m} a_{ij}} \le \boxed{\frac{1}{m}\sum_{j=1}^{m} a_{ij}^{m}}$$

$\mathrm{GM}(a_{i1}^m, \ldots, a_{im}^m)$ $\quad$ $\mathrm{AM}(a_{i1}^m, \ldots, a_{im}^m)$

Guide:

1. '$\sum_i \boxed{\prod_j a_{ij}}$' in LHS of previous slide seems hard, so tackle each row product $\boxed{\prod_j a_{ij}}$ with AM–GM first. A "product" reminds us of "geometric mean".
2. The power 'm' (in superscript) and the '$\sum$' in RHS of previous slide seems to be an arithmetic mean.

Proof step 3: make LHS appear

In target LHS, we have '$\sum_i$' on the left of '$\prod_j$', so take '$\sum_i$' on both sides of the inequality in the previous step.

$$\sum_{i=1}^{n} \prod_{j=1}^{m} a_{ij} \leq \sum_{i=1}^{n} \frac{1}{m} \sum_{j=1}^{m} a_{ij}^{m}$$

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$$= \frac{1}{m}\sum_{j=1}^{m} \|\boldsymbol{a}_j\|_m^m$$

each column is normalized

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$$= 1$$

Equality case

We've applied the AM–GM inequality to each row product, so equality holds if and only if all entries in each row are equal,
i.e. $a_{i1} = \cdots = a_{im}$ for all (row index) $i \in \{1, \ldots, n\}$.

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i.e. $a_{i1} = \cdots = a_{im}$ for all (row index) $i \in \{1, \ldots, n\}$.
In this case, each corresponding component in any two distinct column vectors is equal.
i.e. for all (row index) $i \in \{1, \ldots, n\}$ and (column indices) $j, j' \in \{1, \ldots, m\}$, $a_{ij} = a_{ij'}$.

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Two vectors are equal to each other if and only if each of their corresponding components are equal.

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Since we have normalized each column in the first step of our proof, we have **equality holds if and only if all column vectors in the matrix are parallel to each other**.

Corollary: Carlson's inequality

$$\frac{\sum_{i=1}^{n} \sqrt[m]{\prod_{j=1}^{m} a_{ij}}}{n} \leq \sqrt[m]{\prod_{j=1}^{m} \frac{\sum_{i=1}^{n} a_{ij}}{n}}$$

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Proof. Apply the generalized Cauchy–Schwarz inequality to the matrix

$$\begin{bmatrix} a_{11}^{1/m}/n & \cdots & a_{1m}^{1/m}/n \\ \vdots & \ddots & \vdots \\ a_{n1}^{1/m}/n & \cdots & a_{nm}^{1/m}/n \end{bmatrix}.$$

□

Application: avoid fractional powers

The figure in the slide for the generalized Cauchy–Schwarz inequality is often too difficult to apply on questions. In practice, we often take the m -th power on both sides to avoid fractional powers.

$$\text{i.e. } \left(\sum_{i=1}^{n}\prod_{j=1}^{m} a_{ij}\right)^{m} \leq \prod_{j=1}^{m}\left(\sum_{i=1}^{n} a_{ij}^{m}\right).$$

The following question is a good example to illustrate how arranging terms in the form of a matrix can help organizing thoughts.

Example: optimization of sum of reciprocals

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Discussion:

1. Constraint: Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$.
2. Objective function should stay on RHS.
3. Tricky part:

$$\begin{bmatrix} \frac{2}{\sin\theta} & \sin^2\theta \\ \frac{3}{\cos\theta} & \cos^2\theta \end{bmatrix}.$$

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Quickfix: adjust the power of each term by taking the m-th root

i.e. $$\begin{bmatrix} \left(\frac{2}{\sin\theta}\right)^{1/3} & \left(\frac{2}{\sin\theta}\right)^{1/3} & \sin^{2/3}\theta \\ \left(\frac{3}{\cos\theta}\right)^{1/3} & \left(\frac{3}{\cos\theta}\right)^{1/3} & \cos^{2/3}\theta \end{bmatrix}.$$

Problem solving flow for minimization problems

It would be hard to get the final matrix at the first sight, so I suggest the following steps (to "get the row product right" first).

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6. Apply the generalized Cauchy–Schwarz inequality to the matrix.

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6. Apply the generalized Cauchy–Schwarz inequality to the matrix.
7. State the equality case.

Practice: generalization of previous example

Exercise. If $a, b > 0$, $n \in \mathbb{N}$, $0 < \theta < \pi/2$, show that

$$\left(a^{2/(n+2)} + b^{2/(n+2)}\right)^{(n+2)2} \leq \frac{a}{\sin^n \theta} + \frac{b}{\cos^n \theta}.$$

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Hint:

- two columns of $\begin{bmatrix} a/\sin^n \theta \\ b/\cos^n \theta \end{bmatrix}$
- n columns of $\begin{bmatrix} \cos^2 \theta \\ \sin^2 \theta \end{bmatrix}$

Practice: Power Mean Inequality for integer power

Exercise. For any positive real numbers $a_1, \ldots, a_n$ and positve integer $p > 0$, show that

$$\left(\frac{\sum_{i=1}^{n} a_i^p}{n}\right)^{1/p} \geq \frac{\sum_{i=1}^{n} a_i}{n}.$$

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Solution. Apply the generalized Cauchy–Schwarz inequality to the matrix

$$\begin{bmatrix} a_1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ a_n & 1 & \cdots & 1 \end{bmatrix}$$

with $p - 1$ columns of 1's.

Variation: Generalized Titu's Lemma

For any real numbers $a_1, ..., a_n$, positive real numbers $b_1, ..., b_n$, positive integers $m, k \in \mathbb{N}$ such that $k > m$,

$$\sum_{i=1}^{n} \frac{a_i^k}{b_i^m} \geq n^{1+m-k} \frac{\left(\sum_{i=1}^{n} a_i\right)^k}{\left(\sum_{i=1}^{n} b_i\right)^m}.$$

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Proof. Focus on the "draft matrix"

$$\begin{bmatrix} a_1^k/b_1^m & b_1 & \cdots & b_1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_n^k/b_n^m & b_n & \cdots & b_n & 1 & \cdots & 1 \end{bmatrix}$$

with m columns of b's and $k - 1 - m$ columns of 1's. □

Last example

Most other questions are direct consequences of the previous lemma, including the following:

Exercise. For any a, b, $c > 0$ satisfying $abc = 1$, and positive $k \geq 2$, show that

$$\sum_{\text{cyc}} \frac{1}{a^k(b+c)} \geq \frac{3}{2}.$$

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Attempt: Replace the numerator on LHS by abc. Then

$$\text{LHS} = \sum_{\text{cyc}} \frac{\left(\frac{1}{a}\right)^{k-1}}{\frac{1}{b}+\frac{1}{c}} \geq \frac{\left(\sum_{\text{cyc}} \frac{1}{a}\right)^{k-1}}{2\sum_{\text{cyc}} \frac{1}{a}} = \frac{1}{2} \cdot 3^{1+1-(k-1)} \cdot \left(\sum_{\text{cyc}} \frac{1}{a}\right)^{k-2} \underset{\uparrow}{\geq} \frac{1}{2} \cdot 3^{(3-k)+(k-2)}$$

AM–GM inequality

Problem: To apply the generalized Cauchy–Schwarz inequality, we need $k - 1 > 1$, i.e. $k > 2$. The author of the original article doesn't address the case when $k = 2$,

Last example (continued)

which turns out to be the Nesbitt's inequality:
For any positive real numbers a, b, c, we have

$$\sum_{\text{cyc}} \frac{a}{b+c} \geq \frac{3}{2}.$$

Observe that the above inequality is homogeneous, so WLOG, we can assume $a + b + c = 1$. Then it's equivalent to

$$\sum_{\text{cyc}} \frac{a+b+c}{b+c} \geq \frac{3}{2} + 3.$$

The numerator on LHS is $1 = 1^2$, so that Titu's Lemma can be used.